A Framework for the Discussion of Singularities in General Relativity

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The occurrence and nature of singularities in General Relativity remains arguably the most important outstanding problem in the field. This is in direct contrast to most other areas of physics where singularities are merely mathematical peculiarities. We examine why singularities in General Relativity are different—they have a complex and subtle nature and are notoriously difficult to investigate due to the lack of a suitable framework. We consider the effectiveness of various attempts since the '60s to devise such a framework or "boundary construction" for the study of space-time singularities. Finally, we discuss the most recent such boundary construction, the a-boundary; the most objective, flexible and ultimately the most practical of all these constructions. The a-boundary has recently been recast in terms of "distances" rather than its original topological description, rendering it accessible to a much broader range of researchers.

If we put the general question to physicists, "*How should singularities be discussed in physics ?*", most physicists would reply, "*Why should we discuss singularities at all ?*". Indeed, in almost all areas of physics singularities are just mathematical peculiarities; simply the results of incomplete theories. The situation is somewhat more complicated in General Relativity, however. The importance of singularities in General Relativity is demonstrated by two of the most important outstanding problems in the field: the properties of Penrose-Hawking singularities* and the Cosmic Censorship Conjecture. It is well known that the implications of the Cosmic Censorship Conjecture extend beyond General Relativity; it is less well known that the existence of Penrose-Hawking singularities in the quantum realm is still debated.

This establishes the need to ponder and discuss singularities, but gives no indication of a possible framework for our thoughts and discussions on this matter. Whilst in most areas of physics singularities are rather simple, this is certainly not the case in General Relativity. The reason for this goes to the very heart of both the beauty and complexity of General Relativity, and illustrates one of the reasons why General Relativity does not *quite* fit with the rest of physics.

In most areas of physics the theory is constructed in two parts: a space-time†, and an associated mathematical object or objects (e.g. a field, scalar, twistor, etc.). In theories such as these there are only two places that a "singularity" can occur: either in the metric of the space-time or in the associated object(s). Outside of General Relativity the singularity almost always occurs in the associated object(s), due to the assumptions of the theory. In General Relativity, however, the singularity

*That is, those singularities whose existence can be predicted from a singularity theorem.
†In this context, by "space-time" we mean the manifold and metric structure most appropriate for the area of physics being considered.

theorems demonstrate that, under very general physical circumstances, singularities will always occur in the metric. These singularities are fundamentally different from those which, in other areas of physics, only occur in the associated object(s).

This difference can be illustrated by the two most fundamental questions relating to the study of singularities in General Relativity, "*Where is it ?*" (location) and "*What is it ?*" (definition). For singularities only occurring in an associated object, both these questions can readily be answered by simply examining the associated object with reference to its place in the underlying space-time. For metric singularities occurring in General Relativity, however, there is no pre-determined "place" at which to examine them since they do not lie in the given space-time. We must somehow analyse the singularity from within the existing manifold structure. These two problems of location and definition encapsulate the main issues needing to be dealt with when considering singularities in General Relativity. The fact that these problems were discussed at length in the '60s[1] and that papers on the subject are still being produced today[2] demonstrate that these two problems are indeed of a deep, highly complex and subtle nature.

The area of physics described as "Boundary Constructions" is dedicated to the study and potential resolution of these two problems. The idea is that one devises a boundary construction by determining a method of adding a boundary to the space-time under consideration. With the additional presence of the boundary, one then attempts, firstly, to answer the question of the location of the metric singularities, and then to answer the question of their definition.

There are a variety of boundary constructions in existence: the g-boundary,[3] b-boundary,[4] and c-boundary,[5] for example. Each boundary construction attempts to "fix" the location of the singularities, so that in different coordinate patches the singularity always "looks" the same. For example, if in one set of coordinates the singularity is a point, then in all other sets of coordinates the singularity is also a point.

It is interesting to note, however, that during the years since their first introduction, it has been demonstrated that all of these constructions have their failings and that all of their failings can be traced back to one problem; a rigidity of approach that forces these boundaries to contain an element of subjectivity when being applied. This subjectivity means that these boundary constructions fail to give consistent definitions of singularities across multiple space-times and that, potentially, when different relativists apply the same boundary construction to the same space-time they can get different answers. So far, it has been relativists' common understanding of the properties of the particular space-times being considered, and the fact that most relativists have attempted to apply the boundary uniformly to different space-times, which have prevented this subjectivity from coming to light.

Nonetheless, why use a faulty tool when there is an alternative? And there is an alternative, namely the a-boundary[6] or *abstract boundary* construction. The a-boundary avoids this subjectivity issue with a trade-off. The cost of an objective boundary is that the representation of singularities in different coordinate patches may be different. At first glance this state of affairs may seem to cause more problems than the previous approaches. Indeed, it is remarkable that Scott and Szekeres managed to salvage anything at all, let alone a comprehensive and accessible mathematical structure from such a situation.

The *a*-boundary is framed in topological language. Since topology is an area of mathematics that not too many physicists know in detail, the *a*-boundary can appear complex and daunting when first encountered. In turn, this has meant that its many benefits have often been ignored, an example of which is an important theorem due to Ashley and Scott that links the Penrose-Hawking singularity theorems to the existence of essential singularities[‡]. This is the first theorem which has demonstrated that the Penrose-Hawking singularity theorems actually produce singularities which are "real".

Recently the current authors have found that it is possible to frame the *a*-boundary in terms of distances rather than topology[§]. This recasting of the structure into the tangible concept of "distance" brings the *a*-boundary firmly back into the realm of physicists and physics. We hope that, as a result, many more physicists will take an interest in the *a*-boundary and the appropriate framework it provides for the examination of singularities, in a way that yields consistent definitions and intuitive application.

References

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[‡] An essential singularity is a singularity that cannot be removed by an extension of the manifold or by a change of coordinates.[6]

[§] This work is not yet published.